APPLICATIONS OF KNIGHT GRAPHS IN CHESS BOARD

*Dr. Akkili Naresh, Dr.M. Manjula Devi, Prof. G. Shobhalatha**

Department of Mathematics, Sri Krishnadevaraya University, Anantapuramu,

Andhra Pradesh, India-515003

Email: akkilinaresh@gmail.com, manjulamaths.m@gmail.com, gshobhalatha@gmail.com

Abstract: Chess is an ancient game in India. Chess is a war game played between two players. The two players begin the game with the same pieces and settings. So that players must be use tactics and strategy to coin. In this model each player has two Knights and they begin the game on the squarer between their Rooks and Bishops. A knight graph moves two spaces over and one space to the side(L). In this paper the graphical representation of the nxn knight graph discussed by using the adjacency matrix of knight graph. Then the resultant graphs are complete regular graphs.

Key Words: Chess, Knight Graph, adjacency matrix, complete regular graph.

Introduction:

Chess is an ancient game that began in 6^{th} century India and spread through out the world as it evolved into the form we know today. Game pieces found in Russia, China, India, Central Asia Pakistan, and elsewhere that have been determined to be older than that are now regarded as coming from earlier distantly related board games, often involving dice and sometimes using playing boards of 100 or more squares[1].

Modern chess developed during the middle ages. As chess became more popular in Europe, so did Chess-based puzzles. Watkins writes that the earliest Chess board puzzle he knows of is Guarini's problem from 1512.

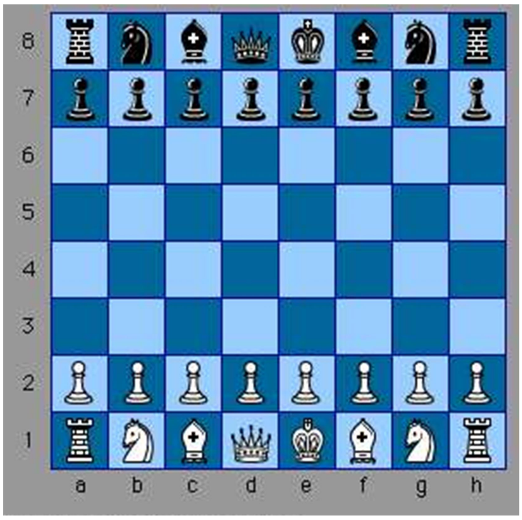
Several famous Mathematicians have worked on various chess puzzles, including Leon Hard Euler and Carl Friedrich Gauss who worked on the Knight's tour and Eight Queen problem, respectively. Queens domination is another popular problem. Problem about the queen piece are well-studied and have been expanded into the other chess pieces. Problems on an $n \times n$ board have thoroughly considered. Mathematicians and Computer scientists have used computer algorithms to resolve chess problems since the mid-twentieth century[2].

Chess is a war game played between two players. The two players begin the game with the same pieces and settings so that the players must be use tactics and strategy to coin.

Chessboard layout :

A standard chessboard is the 8×8 checkerboard. This board consists of eight rows and eight columns made up of an equal number of alternating black and white squares. There are 32 chess pieces covering half of the standard 64 square of each color set , there one King , one Rook, two Bishops, two Knights, two Rooks and eight Pawns.

At the beginning of a standard game, each player lays out his/her pieces as shown fig.



Moves:

The board represents a battlefield in which two armies fight to capture each other‘s king. A player’s army consists of 16 pieces that begin play on the two ranks closest to that player. There are six different types of pieces: King, Rook, Bishop, Knight, Rooks and pawn; the pieces are distinguished by appearance and by how they move. The players alternate moves, white going first.

King: A King can move one square on any direction horiontically, vertically or diagonally.

Queen: The Queen is the most powerful pieces in the game of chess, able to move any no of squares vertically, horizontically or diagonally. Each player starts the game with one queens placed on the middle of the first ran next to the king .Because the queen is the strongest piece in the chessboard.

Bishop: The Bishop can move in any direction diagonally, so long as it is not obstructed, by another piece. The bishop piece cannot move past any piece that is obstructing its path. The bishop can take any other piece on the board that is within its bounds of movement. The bishop has no restriction on distance for each move, but is committed to diagonal movement.

Rook: The Rook moves horizontally or vertically, through any number of unoccupied squares. Each side starts with two rooks, one on the queen side and one on the king side .All some rooks-are located on the corners of the boards.

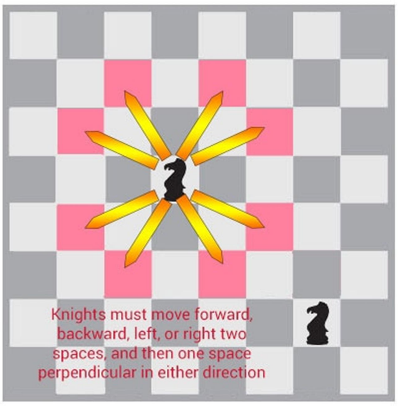
Pawn: When a game begins each side starts with eight pawns. White's pawns are located on the second rank, while Black's pawns on located on the seventh rank. The pawns is the leas powerful piece and is worth on it. If it is pawns first move, it can move forward one or two squares.

Now we discuss only knight graph. We determine some graph theoretic problems on the night tour where we describe how a knight chess piece move. The knight chess piece moves in a very mysterious way. Unlike Rooks, Bishops or Queens, the Knight is limited in the number of squares it can move across. In fact, its movement is a very specific movement. The piece moves in a shape similar to the uppercase "L". Here are the specifics:

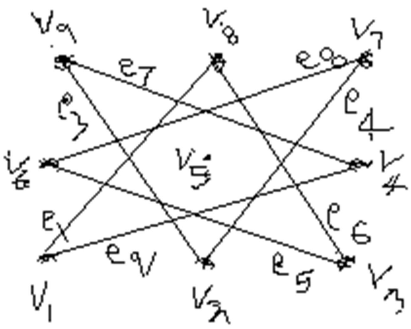
Knight : **Each player has two knights, and they begin the game on the squares between their rooks and bishops. A knight graph moves two spaces over and one space to the side (L)**[2].

- The knight piece can move forward, backward, left or right two squares and must then move one square in either perpendicular direction.
- The knight piece can only move to one of up to eight positions on the board.

- The knight piece can move to any position not already inhabited by another piece of the same color.
- The knight piece can skip over any other pieces to reach its destination position.



3×3 **Knight graph possibilities(L) :**



Incidence matrix : Let G be a graph with n vertices , e edges , and no self loops. Define an $n \times e$ matrix $X[x_{ij}]$, n rows correspond to the n-vertices and the e columns correspond to the e edges as follows:

$$x_{ij} = \begin{cases} 1 \ ; \ if \ \ j^{th} \ edge \ e_j \ is \ incident \ on \ i^{th} \ vertex \ v_i \ , \ and \\ 0 \ ; \ if \ \ otherwise \end{cases}$$

Such a matrix X is called the vertex-edge incidence matrix, or simply incidence matrix. Matrix X for a graph G is sometimes also written as X[G].The incidence matrix contains only two elements $0\ and\ 1$ such a matrix is called a binary matrix or a $(0\ \ 1)$ - matrix. Let us stipulate that these two elements are from Galois field modulo 2[3-4].

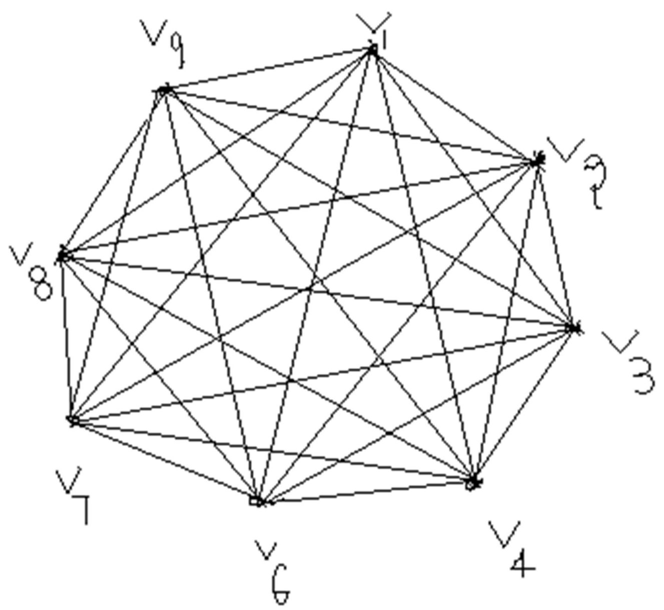
$$X(G)=\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}_{9\times 8}$$

Therefore the row represents to the vertices that is $v_1 v_2 \ldots\ldots v_9$ and the column represents to the edges that is $e_1 e_2 \ldots\ldots e_8$

The degrees of above matrix:

$$d(v_1) = d(v_2) = d(v_3) = d(v_4) = d(v_6) = d(v_7) = d(v_8) = d(v_9) = 2$$

If two vertices are adjacent iff they have same degree i.e. $d(v\ \ ) = 2$



$k_8$complete7-regular graph

Note: 1. A row with all $0's$, therefore , represents an isolated vertex.

2. Circuit matrix does not exist to 3×3 Knight graph.

Path matrix : from 3 × 3 knight graph between (v_1, v_2).

So, There are two different paths $\{e_1, e_6, e_5, e_8, e_4\}$, $\{e_2, e_7, e_3\}$.

$$p(v_1, v_2) = \begin{array}{c} \begin{matrix} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

Adjacency matrix :

As an alternative to the incidence matrix, it is some times more convenient to represent a graph by its adjacency matrix or connection matrix. The adjacency matrix of a graph G with n-vertices and no parallel edges in an $n \times n$ symmetric binary matrix $A = \lfloor a_{ij} \rfloor$ defined by over the ring of integers such that[4].

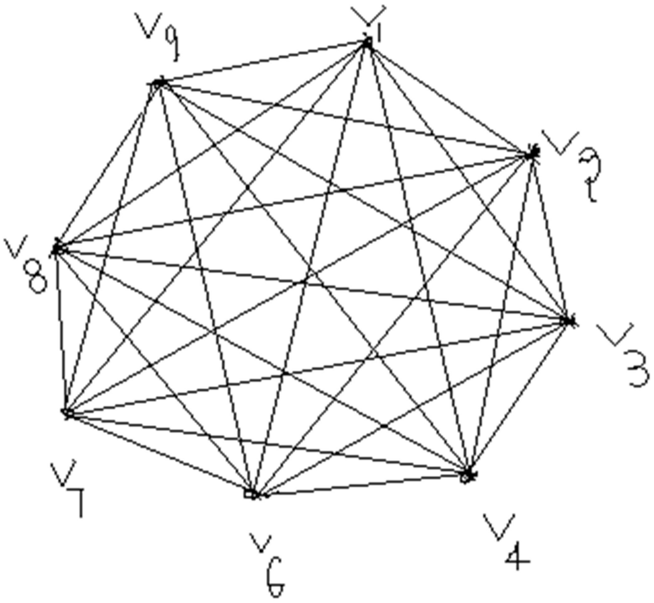
$$a_{ij} = \begin{cases} 1 \ ; \ \textit{if there is an edge between } i^{th} \ \& \ j^{th} \textit{ vertices, and} \\ 0 \ ; \ \textit{if there is no edge between them.} \end{cases}$$

$$A[G] = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}_{9 \times 9}$$

Here the rows or columns represents to the vertices, that is $v_1 v_2 \ldots\ldots. v_9$

$d(v_1) = 2$, $d(v_2) = 2$, $d(v_3) = 2$, $d(v_4) = 2$, $d(v_6) = 2$, $d(v_7) = 2$, $d(v_8) = 2$, $d(v_9) = 2$.

If two vertices are adjacent iff they have same degree i.e. $d(v\ ) = 2$



$k_8$complete7-regular graph

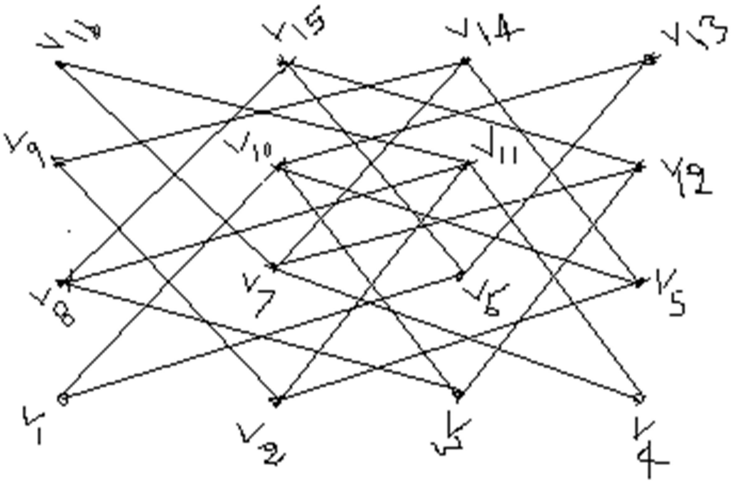
Note: Above matrix is a symmetric matrix i.e. $A^T = A$.

Properties:

1. The entries along the principal diagonal of A are all $0's$ iff the graph has no self loops: A self loop at the i^{th} vertex corresponds to $a_{ii} = 1$.

2. Eigen values of symmetric matrix are real and distict.

4×4 **Knight graph :**



The above figure edges are:

$e_1 = (v_1, v_{10}), e_2 = (v_1, v_6), e_3 = (v_2, v_9), e_4 = (v_2, v_{11}), e_5 = (v_2, v_5)$
$e_6 = (v_3, v_8), e_7 = (v_3, v_{10}), e_8 = (v_3, v_{12}), e_9 = (v_4, v_7), e_{10} = (v_4, v_{11}),$
$e_{11} = (v_5, v_{10})$, $e_{12} = (v_5, v_{14})$, $e_{13} = (v_6, v_{15})$, $e_{14} = (v_6, v_{13})$, $e_{15} = (v_7, v_{12})$, $e_{16} = (v_7, v_{14})$, $e_{17} = (v_7, v_{16}), e_{18} = (v_8, v_{11})$, $e_{19} = (v_8, v_{15}), e_{20} = (v_9, v_{14}), e_{21} = (v_{10}, v_{13})$, $e_{22} = (v_{11}, v_{16})$, $e_{23} = (v_{12}, v_{15})$.

Incidence matrix of 4×4 Knight graph :

$$x_{ij} = \begin{cases} 1 \ ; \ if \ \ j^{th} \ edge \ e_j \ is \ incident \ on \ i^{th} \ vertex \ v_i \, , \ and \\ 0 \ ; \ if \ \ otherwise \end{cases}$$

$$\begin{bmatrix}
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix}$$

Therefore the row represents to the vertices that is $v_1 v_2 \ldots\ldots v_{16}$ and the column represents to the edges that is $e_1 e_2 \ldots\ldots e_{23}$

Note: Above matrix is not symmetry.

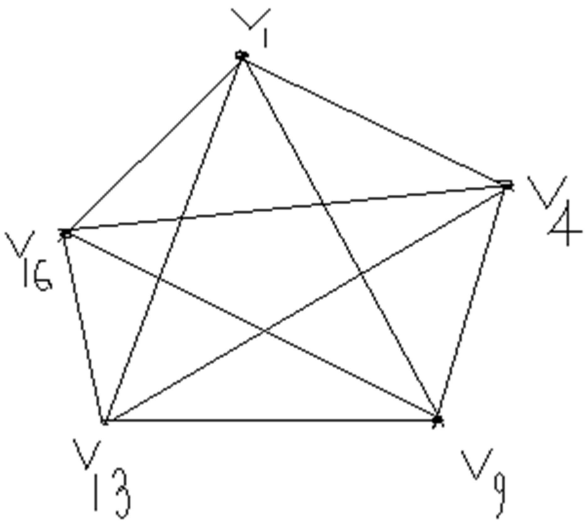
Now we write degrees of above matrix are

d($v_1) = 2$, $d(v_2) = 3$, $d(v_3) = 3$, $d(v_4) = 2$, $d(v_5) = 3$, $d(v_6) = 3$, $d(v_7) = 4$, $d(v_8) = 3$, $d(v_9) = 2$, $d(v_{10}) = 4$, $d(v_{11}) = 4$,

$d(v_{12}) = 3$, $d(v_{13}) = 2$, $d(v_{14}) = 3$, $d(v_{15}) = 3$, $d(v_{16}) = 2$.

(1) If two vertices are adjacent iff they have same degree i.e. $d(v\ ) = 2$

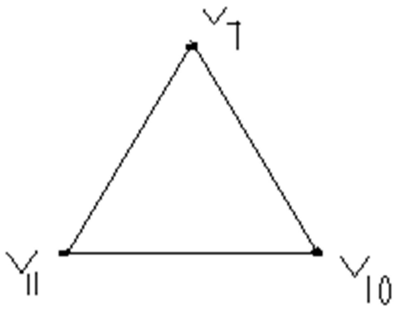
i.e. d$(v_1) = 2$, $d(v_4) = 2$, $d(v_9) = 2$, $d(v_{13}) = 2$, $d(v_{16}) = 2$.



k_5Complete4-regular graph

(2) If two vertices are adjacent iff they have same degree i.e. $d(v) = 4$.

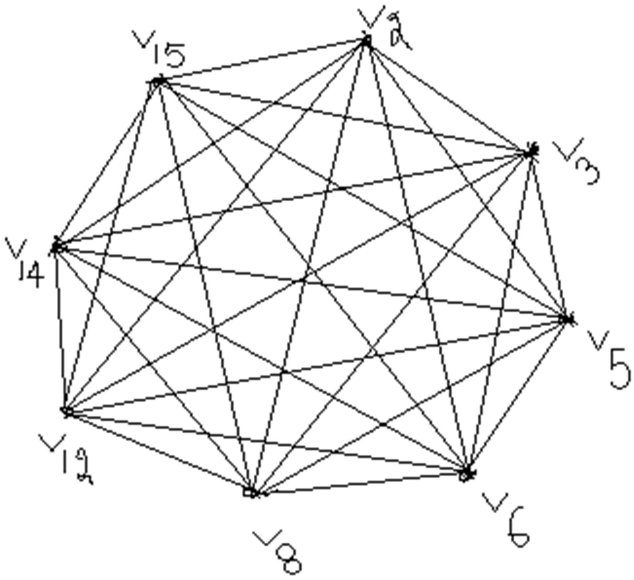
i.e $d(v_7) = 4$, $d(v_{10}) = 4, d(v_{11}) = 4$.



k_3Complete2-regular graph

Similarly, $d(v) = 3$. Those are

$d(v_2) = 3, d(v_3) = 3, d(v_5) = 3, d(v_6) = 3, d(v_8) = 3, d(v_{12}) = 3,$
$d(v_{14}) = 3, d(v_{15}) = 3.$



$k_8$complete7-regular graph

Adjacency matrix of 4×4 **Knight graph :**

$$a_{ij} = \begin{cases} 1 \; ; \; if \;\; there \; is \; an \; edge \; between \; i^{th} \; \& \; j^{th} \; vertices, \;\; and \\ 0 \; ; \;\; if \;\;\; there \; is \; no \; edge \; between \; them. \end{cases}$$

$$A[G]=\begin{bmatrix}
0&0&0&0&0&1&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&1&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&1&0&1&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&1&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&1&0&0&0&1&0&0\\
1&0&0&0&0&0&0&0&1&0&0&0&1&0&1&0\\
0&0&0&1&0&0&0&0&0&0&0&1&0&1&0&1\\
0&0&1&0&0&0&0&0&0&0&1&0&0&0&1&0\\
0&1&0&0&0&1&0&0&0&0&0&0&0&1&0&0\\
1&0&1&0&1&0&0&0&0&0&0&0&1&0&0&0\\
0&1&0&1&0&0&0&1&0&0&0&0&0&0&0&1\\
0&0&1&0&0&0&1&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&1&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&1&0&1&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&1&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&1&0&0&0&0&0
\end{bmatrix}_{16\times 16}$$

$d(v_1)=2$, $d(v_2)=3$, $d(v_3)=3$, $d(v_4)=2$, $d(v_5)=3$, $d(v_6)=4$, $d(v_7)=4$, $d(v_8)=3$, $d(v_9)=3$, $d(v_{10})=4$, $d(v_{11})=4$, $d(v_{12})=3$, $d(v_{13})=2$, $d(v_{14})=3$, $d(v_{15})=3$, $d(v_{16})=2$

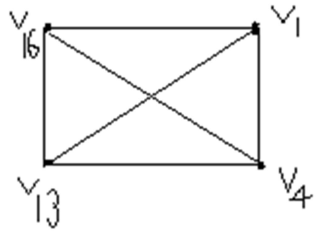
Note: Above matrix is symmetric. i.e. $A^T = A$.

Properties:

1. The entries along the principal diagonal of A are all $0's$ iff the graph has no self loops: A self loop at the i^{th} vertex corresponds to $a_{ii}=1$.
2. Eigen values of symmetric matrix are real and distinct .
3. In 4×4 Knight graph containing squares.
4. Each square contains circuit.

(1) If two vertices are adjacent iff they have same degree i.e. $d(v\ )=2$

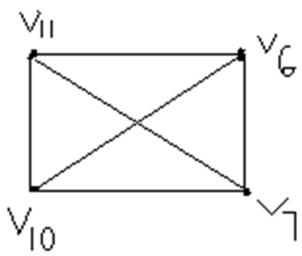
i.e. $d(v_1)=2$, $d(v_4)=2$, $d(v_{13})=2$, $d(v_{16})=2$.



k_4-complete3- regular graph

(2) If two vertices are adjacent iff they have same degree i.e. $d(v\ \ ) = 4$

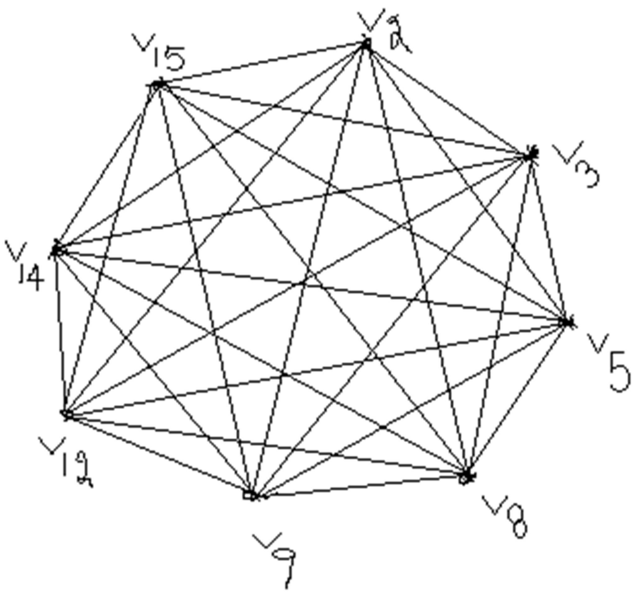
i.e. $d(v_6) = 4$, $d(v_7) = 4$, $d(v_{10}) = 4$, $d(v_{11}) = 4$.



k_4-complete3- regular graph

(3) If two vertices are adjacent iff they have same degree i.e. $d(v\ \ ) = 3$

i.e. $d(v_2) = 3$, $d(v_3) = 3$, $d(v_5) = 3$, $d(v_8) = 3$, $d(v_9) = 3$, $d(v_{12}) = 3$, $d(v_{14}) = 3$, $d(v_{15}) = 3$.



$k_8$complete7-regular graph

Continuing like this process by remaining knight graphs also complete regular subgraphs.

Theorem : Let $'L'$ be a $n \times n$ knight graph and $G = (V, E)$ denote the graph of knight tour with vertex set V, where the vertices of v are the entries of the adjacency matrix of knight graph and edge set E is defined as follows

$E(G) = \{x, y \in V, x \neq y / x \text{ and } y \text{ are adjacent iff they have same degree}\}$. Then the resultant graphs are complete regular graphs.

Proof: Given that L be a $n \times n$ knight tour

Assume $G = (V, E)$, where V is the vertex set and vertices of v are the entries of the adjacency matrix of knight graph and E is edge set such that

$$E(G) = \{x, y \in V, x \neq y / x \text{ and } y \text{ are adjacent iff they have same degree}\}$$

In knight graph shift, we construct the adjacent matrix, by adjacent matrix we find degrees of each element.

These degrees are chosen vertices and connected to each other.

Now , If $n = 1$ or $n = 2$ then in these two cases it is observed two spaces over and one space to the side(L) does not exist.

If $n = 3$ then in each step moved by the knight graph can be represented as an edge in the graph, i.e. two spaces over and one space to the side (L) exists.

Continuing like this , in all possible knight shift we obtained all the possible edges which will form a graph and this graph are complete regular subgraphs[5-6].

References:

1. Hare, E. and Hedetniemi, S. “A linear algorithm for computing knight’s domination number of a *k x n* chessboard”, Congr. Number., Vol. 59(**1987**),pp115-130.
2. Chia, G.L. “ Generalised Knight’s tours on Rectangular Chessboards(**2005**)”.
3. Jean-Pierre Series. “ Algebraic Groups and Classfields(**1988**)”.
4. Narsingh Deo. “Graph Theory with Applications to Engineering and Computer Science”, Prentice Hall of India (**2004**).
5. Swetha, A. and Shobhalatha, G. “Graphs on commutative rings of types Z_n and $Z_n \times Z_n$”, Int. J. Math. Arch., Vol.7(2) (**2016**),pp.135-141.
6. Swetha, A. and Shobhalatha, G. “Graphs for Direct product of Commutative Rings”. Int. J. Math. Arch., Vol-6(11)(**2015**),pp71-75.