# REGULAR ZERO-DIVISOR GRAPHS ON $(z_n,.)$

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***Abstract:*** *A graph in which all vertices are of equal degree is called a regular graph. A (finite or infinite) complete bipartite graph together with some end vertices all adjacent to a common vertex is called a complete bipartite graph .In this paper it is proved G is the graph of a group with zero if and only if it is one of the following graphs. Complete graph, star graph, two-star graph, complete bipartite graph. It is proved that a non zero-divisor graph (including a+b=0, $a \neq b$) is regular if and only if it contains all possible triangles.*
*In view of the above facts we want to present same regular graphs with different properties.*

***Key Words***: **Zero-divisors, Bipartite graph, regular graph, a complete bipartite graph**.

## 1. INTRODUCTION

In mathematics, graphs are useful on geometry and certain parts of topology, for example knot theory. Algebraic graph theory has close links with group theory. In this paper we discuss the graph theoretic properties on the group $z_n$ .That is throughout this paper, we consider the group $(z_n, +_n)$, where $z_n$ = {0, 1, 2...........} with the binary operation '$+_n$'(addition modulo n). $z_n$ is called the group of integers modulo n. The graph on the group $z_n$ is denoted G= (V, E) and is called the graph of $z_n$ with vertex set V(G) and edge set E(G).The elements of $z_n$ are considered as the vertices of G and edge set of G is defined by assuming condition on the elements of $z_n$ .It is observed that .The resultant graphs obtained on group $z_n$ are regular graphs[1-5].

## 2. BASIC CONCEPTS AND PRELIMINARIES

**Group:** A non empty set G with a binary operation'. 'on G is called a group if the following axioms hold:

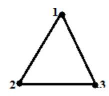
(i) a(bc) = (ab)c for all a,b,c$\in$ G.

(ii) There exists e$\in$ G such that ea=a for all a$\in$ G

(iii) For every a$\in$ G There exists $a'\in$ G such that a $a'$ = e.

**Degree of vertex:** In a graph G ,the number of edges incident on a vertex ' v' is called the degree of vertex ' v' and is denoted by d(v).

**Regular graph:** A graph G is said to be k-regular, if d (v)=k, for some positive integer k and for every v in V(G).A regular graph is one that is k-regular for some positive integer k.




**2-regular graph**.

## 3. GRAPHS AND THEIR ILLUSTRATION ON GROUP $z_n$

This section deals with the graphs obtained on the elements of $z_n$ , which are considered as the vertices of the vertices of the graph G of $z_n$ . The resultant graph obtained is regular graphs or complete graphs or bipartite graphs.

**3.1Theorem:**Let $z_n$ be a group consider the graph G of $z_n$ ,where G=(V,E) such that V is the vertex set and E be the edge set of G of the elements of $z_n$ are considered as the vertex of a graph G and the edge set E of G is defined by E={(a,b)/a is adjacent to b $\Leftrightarrow$ a+b $\neq$ 0 for every a $\neq$ b, such that a,b are two distinct non zero elements in $z_n$ }.Then the graph G is regular graph.

Proof: Let $z_n$ be the group of integers modulo n and G= (V, E) denote the graph of $z_n$ with vertex set V and edge set E. Now, a, b are adjacent $\Leftrightarrow$ a $+_n$ b $\neq$ 0. That is a $+_n$ b= {a+b if a+b<n and 0 if a+b=n}.So, a, b are adjacent $\Leftrightarrow$ a+b<n.

Case (i): If n is odd. Then for any two elements a,b $\in z_n$ where a,b $\neq$ 0 and a+b<n implies a and b are adjacent. Since 'n 'is odd, so there are (n-2) elements are adjacent to every element a $\neq$ 0 in $z_n$ .Then every element of $z_n$ is adjacent (n-2) elements. i.e. every vertex in the resultant graph G of $z_n$ has degree n-2.Therefore the graph G is a (n-2)-regular graph.

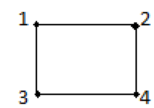
Case (ii): If 'n' is even. Then for any two elements, a,b $\in z_n$ where a,b $\neq$ 0 and a+b<n implies a and b are adjacent. Since 'n' is even, so in $z_n$ there must exist exact one element 'x' which has self inverse i.e. x

$+_n$ x=0 and x+y $\neq 0$ where $\forall$ y $\in z_n$ and y $\neq$ x, x is not adjacent to x. So,x is adjacent to the remaining (n-2) elements. The elements other than x of $z_n$ have different inverses and so every element other than 'x' is adjacent to (n-3) elements. Hence, in the graph G of $z_n$ are of different degrees. Therefore the graph is not regular.

**Illustration:** Let $z_n$ be a group of integers modulo n satisfying all the condition of the above theorem.

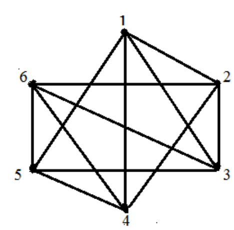
**Case (i):** If $r=0$ then $n=1$, only one vertex. Therefore the graph G has null graph.

**Case(ii):** Let n=5 then $z_5 = \{0,1,2,3,4\}$ V={1,2,3,4} and E={ $e_1, e_2, e_3, e_4$ }where $e_1$=(1,2) $e_2$=(2,4) $e_3$ =(3,4) and $e_4$=(1,3), Since $1+_5 2 \neq 0$ $2+_5 4 \neq 0$ ,$3+_5 4 \neq 0$ and $1+_5 3 \neq 0$. Hence the graph of G is a regular graph[6].



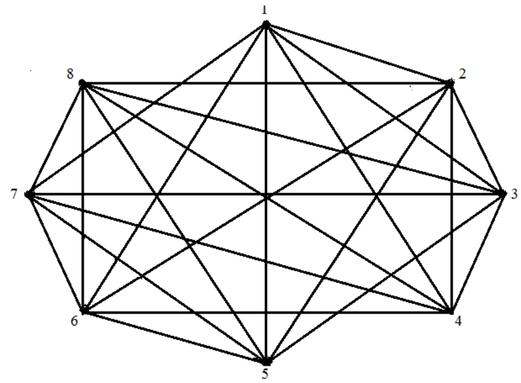

**2-regular graph**

**Case(iii):** Let n=7 then $z_7$={0,1,2,3,4,5,6},V={1,2,3,4,5,6},and E={ $e_1, e_2, e_3, e_4 \ldots\ldots.. e_{12}$ },where $e_1$ =(1,5) $e_2$ =(1,4) $e_3$=(1,3) and $e_4$=(1,2)…………….etc,Since $1+_7 5 \neq 0$ $1+_7 4 \neq 0$ ,$1+_7 3 \neq 0$ and $1+_7 2 \neq 0$ ………..etc. Hence the graph of G is a regular graph.




**4-regular graph**

**Case(iv):** Let n=9 then $z_9$={0,1,2,3,4,5,6,7,8},V={1,2,3,4,5,6,7,8} and E={ $e_1, e_2, e_3, e_4$ $e_5, e_6 \ldots\ldots.. e_{24}$ },where $e_1$=(1,7) , $e_2$ =(1,6), $e_3$=(1,5) , $e_4$ =(1,4), $e_5$=(1,3) , $e_6$=(1,2)…………….etc,Since $1+_9 7 \neq 0$ , $1+_9 6 \neq 0$ ,$1+_9 5 \neq 0$ , $1+_9 4 \neq 0$, $1+_9 3 \neq 0$ , $1+_9 2 \neq 0$ ………..etc. Hence the graph of G is a regular graph.




**6-regulargraph**

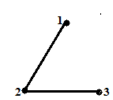
From the following graph, we say that graph G is a 2k-regular graph, continuing like this if 2r+1, where r=2,3,4……..

Then the graph G is a 2, 4, 6……….regular graphs respectively.

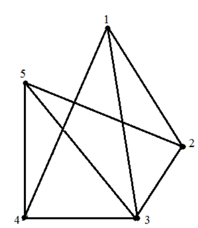
**Result:** If $z_n$ be a group of integers modulo n, and let G= (V, E), where E= {a, b∈ $z_n$/a+b$\neq 0$, for any two distinct adjacent elements a, b$\neq 0$ and a$\neq$b. a,b∈ $z_n$ }. Then the graph G is not a regular graph.

e.g.(1): If n=2, $z_2$={0,1 } is null graph.

e.g.(2):If n=4 then $z_4$={0,1,2,3}, V={1,2,3} and E={ $e_1, e_2$ } where $e_1$=(1,2) , $e_2$ =(2,3), Since $1 +_4 2 \neq 0$, $2 +_4 3 \neq 0$ .Hence the graph of G is not a regular graph.




e.g.(3):If n=6 then $z_6$= {0,1,2,3,4,5}, V={1,2,3,4,5} and E={ $e_1, e_2, e_3$........$e_8$ },where $e_1$=(1,4), $e_2$ =(1,3), $e_3$=(1,2), Since $1 +_6 4 \neq 0$, $1 +_6 3 \neq 0$ ………..etc. Hence the graph of G is not a regular graph.




From the following graph, we say that graph G is not a regular graph, continuing like this if $z_n$, where n=2, 4, 6…….. (even numbers).

**Conclusion:** From the above illustration we observe that every vertex in G has degree k, where k=2r-2, when n=2r+1, r=2, 3……….. and G is k-regular graph. The following table shows the relation between 'n' and 'k ', where n is number of vertices and k is degree.

| r | **Number of vertices n=2r+1(n>3)** | **K=(2r-2)-regular graph** |
|---|---|---|
| 2 | n=5 | 2-regular graph |
| 3 | n=7 | 4-regular graph. |
| 4 | n=9 | 6-regular graph. |
| . | . | . |
| . | . | . |
| . | . | . |

**3.2Corollary:** Let $z_n$ be a group consider the graph G of $z_n$ ,where G=(V,E) such that V is the vertex set and E be the edge set of G of the elements of $z_n$ are considered as the vertex of a graph G and the edge set E of G is defined by E={(a,b)/a is adjacent to b $\Leftrightarrow$ a+b$\neq$0 including a+b=0, for every two distinct non zero elements a,b$\in$ $z_n$ }.Then the graph G is a regular graph.

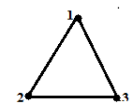
Proof: Let $z_n$ be the group of integers modulo n and G= (V, E) denote the graph of $z_n$ with vertex set V and edge set E. If a, b are adjacent $\Leftrightarrow$ $a +_n b \neq 0$ and $a +_n$ b=0. That is $a +_n$ b= {a+b if a+b<n and 0 if a+b=n}.So, a, b are adjacent $\Leftrightarrow$ a+b<n.

If n is even. Then for any two elements a, b $\in$ $z_n$ where a, b$\neq$0 and a+b<n implies a and b are adjacent. Since 'n 'is even, So there are (n-3) elements are adjacent to every element a$\neq$0 in $z_n$ .That is every element of $z_n$ is adjacent (n-3) elements. i.e. every vertex in the resultant graph G of $z_n$ has degree n-3.Therefore the graph G is a (n-3)-regular graph.

**Illustration:** Let $z_n$ be a group of integers modulo n satisfying all the conditions of the above theorem.

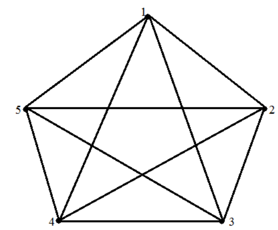
**Case (i):** If n=2 then $z_2$ ={0,1}, V={1} and E={0}. Therefore the graph G has null graph.

**Case(ii):** Let n=4, then $z_4$ = {0,1,2,3}, V={1,2,3} and E={ $e_1, e_2, e_3$ },where $e_1$=(1,2) , $e_2$ =(1,3) , $e_3$=(2,3) and Since $1 +_4 2 \neq 0$ , $1 +_4$ 3=0 and $2 +_4 3 \neq 0$. Hence the graph of G is a regular graph.



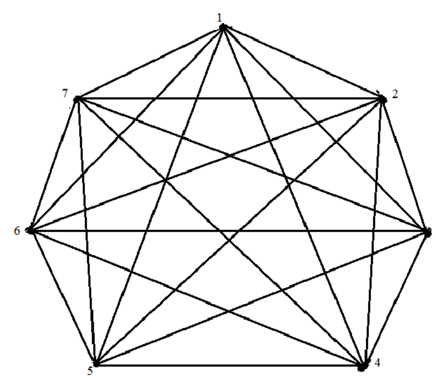

**2-regular graph**

**Case(iii):** Let n=6 then $z_6=\{0,1,2,3,4,5\}$,V={1,2,3,4,5},and E={ $e_1, e_2, e_3, e_4$ …….. $e_{10}$ },where $e_1$=(1,5) , $e_2$=(1,4), $e_3$=(1,3) and $e_4$=(1,2)…………….etc,Since $1+_6 5$=0, $1+_6 4 \neq 0$ ,$1+_6 3 \neq 0$ and $1+_6 2 \neq 0$ ………..etc. Hence the graph of G is a regular graph.




**4-regular graph**

**Case(iv):** Let n=8 then $z_8=\{0,1,2,3,4,5,6,7\}$,V={1,2,3,4,5,6,7} and E={ $e_1, e_2, e_3, e_4$ $e_5, e_6$ …….. $e_{21}$ },where $e_1$=(1,7) , $e_2$=(1,6), $e_3$=(1,5) , $e_4$=(1,4,) $e_5$=(1,3) $e_6$=(1,2)…………….etc,Since $1+_8 7$=0 , $1+_8 6 \neq 0$ ,$1+_8 5 \neq 0$ , $1+_8 4 \neq 0$, $1+_8 3 \neq 0$ , $1+_8 2 \neq 0$ ………..etc. Hence the graph of G is a regular graph.




**6-regular graph**

**Conclusion:** From the above illustration we observe that every vertex in G has degree k, where k=2r-2, when n=2r, r=1,2, 3……….. and G is k-regular graph. The following table shows the relation between 'n' and 'k ', where n is number of vertices and k is degree.

| R | **Number of vertices n=2r** | **K=(2r-2)-regular graph** |
|---|---|---|
| 1 | n=2 | null graph |
| 2 | n=4 | 2-regular graph. |
| 3 | n=6 | 4-regular graph. |
| 4 | n=8 | 6-regular graph |
| . | . | . |
| . | . | . |
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The following result is the immediate consequence of the above illustrations

**Result:** The non zero divisor graph(including a+b=0, $a \neq b$) is regular if and only if it contains all possible triangles.

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