Some results of Multiplicative(Generalized)-Derivations in Semiprime rings

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Abstract

Let R be a semiprime ring. A mapping $F : R \longrightarrow R$ (not necessarily additive) together with a mapping $d : R \longrightarrow R$ (not necessarily a derivation nor additive) is called a multiplicative(generalized)-derivation of R if $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in R$. The main aim of this paper is to study the following conditions: $(i) F(xy) \pm F(x) oy = 0, (ii) F(xy) \pm [F(x), y] = 0, (iii) d(x) F(y) \pm xoy = 0, (iv)$ $d(x) F(y) \pm [x, y] = 0, (v) F(x) oy \pm [x, F(y)] = 0$, for all x, y in some appropriate subsets of R.

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1 Introduction

Through out the present paper R will denote an associative ring with centre $Z(R)$. For any $x, y \in R$, the symbol $[x, y]$ and xoy stands for the commutator $xy - yx$ and anti-commutator $xy + yx$, respectively. A ring R is called prime ring if for any $a, b \in R$, $aRb = (0)$ implies that either $a = 0$ or $b = 0$ and is called semiprime ring if $aRa = (0)$ implies that $a = 0$. For given $x, y \in R$, set $[x, y]_0 = x, [x, y]_1 = [x, y] = xy - yx$ and $[x, y]_k = \left[[x, y]_{k-1}, y\right]$ for $k > 1$. Let S be a non empty subset of R. A map $F : S \longrightarrow R$ is called a centralizing (or commuting) map on S if $[f(x), x] \in Z(R)(\text{or})[f(x), x] = 0)$ for all $x \in S$. An additive map $d : R \longrightarrow R$ is called a derivation of R if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. An additive map $F : R \longrightarrow R$ associated with a derivation $d : R \longrightarrow R$ is called a generalized derivation of R if $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in R$. The notion of generalized derivation was introduced by Bresar[1]. In [5], Hvala gave an algebraic study of generalized derivation in prime rings.

Following [2], a multiplicative derivation of R is a map $d : R \longrightarrow R$ which staitsfies $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. Ofcourse these maps are not additive. The concept of multiplicative derivations appears for the first time in the work of Daif [2] and it was motivated by the work of Martindale [6]. The complete description of these mappings was provided by Goldman and semrl [7]. The notion of multiplicative derivation was extended to multiplicative generalized derivation by Daif and Tammam-El-sayiad [3] as follows: a map $F : R \longrightarrow R$ is called a multiplicative generalized derivation if there exists a derivation d on R such that $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$. Further, Dhara and Ali [4] generalize this definition of multiplicative generalized derivation by considering d as any map from R to R. They defined that a map $F : R \longrightarrow R$ (not necessarily additive) is called multiplicative(generalized)derivation if $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in R$, where $d : R \longrightarrow R$ is any map(not necessarily a derivation nor additive). It is obvious that every generalized derivation is multiplicative(generalized)derivation on R. However, the converse need not be true in general.

During the past few years, some authors have been studying the commutativity in prime and semiprime rings admitting derivations or generalized derivations. Daif and Bell [8], proved that R is semiprime ring, I is a non zero ideal of R and d is a derivation of R such that $d([x, y]) = \pm[x, y]$, for all $x, y \in I$ of R, then R is commutative. This theorems considered for generalized derivations by Quadri et al in [9]. Hongan [10] generalized these results by taking the same situations in the center of the ring R. On the other hand, in [13], Ashraf and Rehman showed that R is prime ring with a non zero ideal U of R and d is a derivation of R such that $d(xy) \pm xy \in Z$, for all $x, y \in U$, then R is commutative. Ashraf et al. proved this result for a generalized derivation of R in [12]. Recently, Dhara and Ali. S [4] studied multiplicative(generalized)-derivations in prime and semiprime rings. Asma Ali et al. [11] investigated the commutativity of a prime ring admitting a generalized derivation staisfying any one of the following identities $(i)F([x,y]) \pm [x,y] \in Z(R)(ii)F(xoy) \pm (xoy) \in Z(R)$ for all x, y in some appropriate subset of R.

In this line of investigation, we extend some results concerning semirpime rings to a multiplicative(generalized)-derivations. In the present paper, our main object is to study some results of multiplicative(generalized)-derivations in semiprime rings. $(i)F(xy) \pm F(x)oy = 0, (ii)F(xy) \pm [F(x), y] = 0, (iii)$ $d(x)F(y) \pm xoy = 0, (iv)d(x)F(y) \pm [x,y] = 0, (v)F(x)oy \pm [x, F(y)] = 0$.

Throughout the paper, we shall frequently use the following basic commutator and anti-commutator identities. For any $x, y, z \in R$. $[x, yz] = y[x,z] + [x,y]z$, $[xy, z] = [x,z]y + x[y,z]$, $xoyz = (xoy)z - y[x,z] = y(xoz) + [x,y]z$, $xyoz = x(yoz) - [x,z]y = (xoz)y + x[y,z]$ for all $x, y, z \in R$.

2 Main Results

Now we begin with our first theorem:

Theorem 2.1. *let R be a semiprime ring and I a nonzero left ideal of R. If $F : R \longrightarrow R$ is a multiplicative(generalized)-derivation associated with the map $d : R \longrightarrow R$ such that $F(xy) \pm F(x)oy = 0$ holds for all $x, y \in I$, then $I[x, d(x)] = 0$ for all $x \in I$.*

Proof. By the assumption, we have

$$F(xy) + F(x)oy = 0, for\ all\ x, y \in I. \quad (2.1)$$

Replacing y by yx in (2.1), we get

$$F(xy)x + xyd(x) + (F(x)oy)x - y[F(x), x] = 0.$$

Using (2.1), we get

$$xyd(x) - y[F(x), x] = 0, for\ all\ x, y \in I. \quad (2.2)$$

Substituting $d(x)y$ for y in (2.2), we get

$$xd(x)yd(x) - d(x)y[F(x), x] = 0, for\ all\ x, y \in I. \quad (2.3)$$

left multiplying (2.2) by $d(x)$, we get

$$d(x)\,xyd(x) - d(x)\,y\,[F(x), x] = 0, for\ all\ x, y \in I. \tag{2.4}$$

Subtracting (2.4) from (2.3), we get

$$[x, d(x)]\,yd(x) = 0, for\ all\ x, y \in I. \tag{2.5}$$

Replacing y by yx in (2.5), we get

$$[x, d(x)]\,yxd(x) = 0, for\ all\ x, y \in I. \tag{2.6}$$

Right multiplying (2.5) by x and subtract it form (2.6) we get

$$[x, d(x)]\,y\,[x, d(x)] = 0, for\ all\ x, y \in I. \tag{2.7}$$

Replacing y by ry, we obtain

$$[x, d(x)]\,ry\,[x, d(x)] = 0, for\ all\ x, y \in I, r \in R. \tag{2.8}$$

Hence $y\,[x, d(x)]\,Ry\,[x, d(x)] = 0$ for all $x, y \in I$. The semiprimeness of R yields that $y\,[x, d(x)] = 0$ for all $x, y \in I$. Therefore, $I\,[x, d(x)] = 0$ for all $x \in I$.
The same argument can be adopted in case $F(xy) - F(x)\,oy = 0$ for all $x, y \in I$. This proves the theorem completely. □

Theorem 2.2. *let R be a semiprime ring and I a nonzero left ideal of R. If $F : R \longrightarrow R$ is a multiplicative (generalized)-derivation associated with the map $d : R \longrightarrow R$ such that $F(xy) \pm [F(x), y] = 0$ for all $x, y \in I$, then $I\,[x, d(x)] = 0$ for all $x \in I$.*

Proof. By the assumption, we have

$$F(xy) - [F(x), y] = 0, for\ all\ x, y \in I. \tag{2.9}$$

Replacing y by yx in (2.9), we get

$$F(xy)\,x + xyd(x) - y\,[F(x), x] - [F(x), y]\,x = 0.$$

Using (2.9), we get

$$xyd(x) - y\,[F(x), x] = 0, for\ all\ x, y \in I. \tag{2.10}$$

substituting $d(x)\,y$ for y in (2.10), we get

$$xd(x)\,yd(x) - d(x)\,y\,[F(x), x] = 0, for\ all\ x, y \in I. \tag{2.11}$$

left multiplying (2.10) by $d(x)$, we get

$$d(x)\,xyd(x) - d(x)\,y\,[F(x), x] = 0, \text{ for all } x, y \in I. \tag{2.12}$$

Subtracting (2.12) from (2.11), we get

$$[x, d(x)]\,yd(x) = 0, \text{ for all } x, y \in I. \tag{2.13}$$

Arguing in the similar manner as in Theorem 2.1, after (2.5) we get the result.
By using similar argument we can get the result for the case $F(xy) + [F(x), y] = 0$ for all $x, y \in I$. □

Corollary 2.3. *Let R be a semiprime ring and $F : R \longrightarrow R$ is a multiplicative (generalized)-derivation associated with a map $d : R \longrightarrow R$. If R satisfies any one of the following identities. $(i).F(xy) \pm F(x)\,oy = 0$, $(ii).F(xy) \pm [F(x), y] = 0$ for all $x, y \in R$, then the map d is a commuting map on R.*

Theorem 2.4. *Let R be a semiprime ring and I a nonzero left ideal of R. If $F : R \longrightarrow R$ is a multiplicative(generalized)-derivation associated with a map $d : R \longrightarrow R$ such that $d(x)\,F(y) \pm xoy = 0$ for all $x, y \in I$, then $I\,[x, d(x)]_2 = 0$ for all $x \in I$.*

Proof. We assume that

$$d(x)\,F(y) + xoy = 0, \text{ for all } x, y \in I. \tag{2.14}$$

Replacing y by yx in (2.14), we obtain

$$d(x)\,F(y)\,x + d(x)\,yd(x) + (xoy)\,x = 0, \text{ for all } x, y \in I.$$

Using (2.14), we get

$$d(x)\,yd(x) = 0, \text{ for all } x, y \in I. \tag{2.15}$$

Substituting $[x, d(x)]\,y$ for y in (2.15), we get

$$d(x)\,[x, d(x)]\,yd(x) = 0, \text{ for all } x, y \in I. \tag{2.16}$$

Left multiplying (2.15) by $[x, d(x)]$ and subtract it from (2.16), we obtain

$$[[x, d(x)], d(x)]\,yd(x) = 0, \text{ for all } x, y \in I. \tag{2.17}$$

Again replacing y by yx in (2.17), we get

$$[[x, d(x)], d(x)]\,yxd(x) = 0, \text{ for all } x, y \in I. \tag{2.18}$$

Right multiplying (2.17) by x and subtract it from (2.18), we get

$$[[x, d(x)], d(x)] y [x, d(x)] = 0, for\ all\ x, y \in I. \quad (2.19)$$

Right multiplying (2.17) by $[x, d(x)]$ we get

$$[[x, d(x)], d(x)] yd(x) [x, d(x)] = 0, for\ all\ x, y \in I. \quad (2.20)$$

Right multiplying (2.19) by $d(x)$ we get

$$[[x, d(x)], d(x)] y [x, d(x)] d(x) = 0, for\ all\ x, y \in I. \quad (2.21)$$

Subtract (2.20) from (2.21) we get

$$[[x, d(x)], d(x)] y [[x, d(x)], d(x)] = 0, for\ all\ x, y \in I. \quad (2.22)$$

Replacing y by ry in (2.22), we obtain

$$[[x, d(x)], d(x)] ry [[x, d(x)], d(x)] = 0, for\ all\ x, y \in I. \quad (2.23)$$

left multiplying (2.23) by y, we get

$$y [[x, d(x)], d(x)] ry [[x, d(x)], d(x)] = 0, for\ all\ x, y \in I. \quad (2.24)$$

The semiprimeness of R implies $y [[x, d(x)], d(x)] = 0$ for all $x, y \in I$. That is $I [x, d(x)]_2 = 0$ for all $x, y \in I$.
Similar proof shows that the same conclusion holds as $d(x) F(y) - xoy = 0$ for all $x, y \in I$ □

Theorem 2.5. *Let R be a semiprime ring and I a non zero left ideal of R. If $F : R \longrightarrow R$ is a multiplicative(generalized)-derivation associated with a map $d : R \longrightarrow R$ such that $d(x) F(y) \pm [x, y] = 0$ for all $x, y \in I$ then $I [x, d(x)]_2 = 0$ for all $x \in I$.*

Proof. We assume that

$$d(x) F(y) - [x, y] = 0, for\ all\ x, y \in I. \quad (2.25)$$

Replacing y by yx in (2.25), we get

$$d(x) F(y) x + d(x) yd(x) - [x, y] x = 0, for\ all\ x, y \in I.$$

Using (2.25), we obtain

$$d(x) yd(x) = 0, for\ all\ x, y \in I. \quad (2.26)$$

Further, the proof follows from Theorem 2.4 after (2.15) we obtain the conclusion.
The same argument can be adopted in case $d(x) F(y) + [x, y] = 0$ for all $x, y \in I$. This proves the theorem completely. □

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Corollary 2.6. *Let R be a semiprime ring and $F : R \longrightarrow R$ is a multiplicative (generalized)-derivation associated with a map $d : R \longrightarrow R$. If $d(x)F(y) \pm xoy = 0$ for all $x, y \in R$, then d is commuting maps on R.*

Proof. By (2.15) in Theorem 2.4, we have $d(x)yd(x) = 0$ for all $x, y \in R$. Therefore $[x, d(x)]y[x, d(x)] = 0$, for all $x, y \in R$. That is, $[x, d(x)]R[x, d(x)] = 0$ for all $x, y \in R$. By semiprimeness of R we get $[x, d(x)] = 0$ for all $x \in R$, thus d is commuting on R. □

Corollary 2.7. *Let R be a semiprime ring and $F : R \longrightarrow R$ is a multiplicative (generalized)-derivation associated with a map $d : R \longrightarrow R$. If $d(x)F(y) \pm [x, y] = 0$ for all $x, y \in R$, then d is commuting maps on R.*

Theorem 2.8. *Let R be a semiprime ring and I a nonzero left ideal of R. If $F : R \longrightarrow R$ is a multiplicative(generalized)-derivation associated with a map $d : R \longrightarrow R$ such that $F(x)oy \pm [x, F(y)] = 0$ for all $x, y \in I$ then $I[x, d(x)] = 0$ for all $x \in I$.*

Proof. By the assumption, we have

$$F(x)oy + [x, F(y)] = 0, for\ all\ x, y \in I. \tag{2.27}$$

Replacing y by yx in (2.27), we get

$$(F(x)oy)x - y[F(x), x] + [x, F(y)]x + y[x, d(x)] + [x, y]d(x) = 0.$$

Using (2.27), we get

$$-y[F(x), x] + y[x, d(x)] + [x, y]d(x) = 0, for\ all\ x, y \in I. \tag{2.28}$$

Replacing y by $d(x)y$ in (2.28), we obtain

$$\begin{aligned} -d(x)y[F(x), x] + d(x)y[x, d(x)] + d(x)[x, y]d(x) \\ + [x, d(x)]yd(x) = 0, for\ all\ x, y \in I. \end{aligned} \tag{2.29}$$

Left multiplying (2.28) by $d(x)$, we get

$$-d(x)y[F(x), x] + d(x)y[x, d(x)] + d(x)[x, y]d(x) = 0, for\ all\ x, y \in I. \tag{2.30}$$

Subtracting (2.30) from (2.29), we get

$$[x, d(x)]yd(x) = 0, for\ all\ x, y \in I. \tag{2.31}$$

Replacing y by yx in (2.31), we get

$$[x, d(x)]yxd(x) = 0, for\ all\ x, y \in I. \tag{2.32}$$

Right multiplying (2.31) by x and subtract it from (2.32), we get

$$[x, d(x)]\, y\, [x, d(x)] = 0, for\ all\ x, y \in I. \tag{2.33}$$

Further, the proof follows from Theorem 2.1, after (2.7) we obtain $I[x, d(x)] = 0$ for all $x \in I$.
Similarly, we can prove for the case $F(x)\, oy - [x, F(y)] = 0$ for all $x, y \in I$.

Corollary 2.9. *Let R be a semiprime ring and $F : R \longrightarrow R$ is a multipliative (generalized)-derivation associated with the maps $d : R \longrightarrow R$ respectively. If $F(x)\, oy \pm [x, d(x)] = 0$ for all $x, y \in R$, then d is commuting maps on R.*

Example1. Consider $R=\left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} \text{ / a, b, c} \in \text{S} \right\}$, where S be a ring. We define maps F, d: R⟶R by F$\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$,
d$\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$,
it is verified that F is a multiplicative(generalized)-derivation associated with a map d. It is easy to see that the identities $(i)F(xy) \pm F(x)\, oy = 0, (ii)d(x)\, F(y) \pm xoy = 0, (iii)F(x)\, oy \pm [x, F(y)] = 0$. If satisfied for all $x, y \in R$. Here R is not semiprime ring because $\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ R $\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = (0)$. Note that d is not commuting maps on R. Hence, the semiprimeness in corollary2.6 is crucial. □

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